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THE HIGHER LOWER CENTRAL SERIES

BY

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ABSTRACT

The Baer invariants $\Gamma_n(G)$ of a group are central extensions of the elements $\gamma_n(G)$ of the lower central series. We show that the inclusions $\gamma_{n+1} \subset \gamma_n$ can be lifted to functor morphisms $\Gamma_{n+1} \to \Gamma_n$ and a canonical Lie algebra, analogous to Lazard's Lie algebra, can be constructed which is explicitly computable. This is applied in various ways.

The lower central series of a group is defined by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ where the square brackets denote the commutator. It is a descending filtration of G and the associated graded object, $\bigoplus_{n\geq 1}\gamma_n(G)/\gamma_{n+1}(G)$, is a Lie algebra, as is well known [7]. The Lie structure comes from the commutator operation in G.

In 1945, Baer [1] described an infinite family of 'invariants' of the group G. Starting from a free presentation of G, i.e. an exact sequence

$$1 \to R \to F \to G \to 1$$

in which F is a free group, Baer showed that $\gamma_n(F)/[R,(n-1)F]$ and $(R \cap \gamma_n(F))/[R,(n-1)F]$ $(n \geq 1)$ are invariants of G. Here $[A, mB]$ denotes A if $m = 0$ and $[[A,(m-1)B],B]$ if $m > 0$. Baer's paper was his (remarkable!) development of Hopf's discovery, in 1942, that $R \cap [F,F]/[R,F]$ is an invariant of G. (Nowadays, this is "Hopf's formula" $R \cap [F,F]/[R,F] \approx H_2(G,\mathbf{Z})$.)

Some years ago, I discovered an infinite sequence of *functors*, $\Gamma_n(G)$, which I thought of as generalizations of the concept of 'universal central extension' of a perfect group. Each $\Gamma_n(G)$ is a *canonical* central extension of $\gamma_n(G)$ and if G is perfect, all these extensions are equal to the universal central extension. Only later

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did I learn of Baer's old paper and realize that my 'functors' were Baer's 'invariants'. (I learnt this from Beno Eckmann who, in turn, knew of Baer's paper from Hopf himself.) However, I pursued my approach, and found that there are some things that Baer missed (or could not be expected to get working, as he did, in pre-Lazard times). The main point of this paper is the existence, for each $m \geq n$, of a natural morphism $\Gamma_m \to \Gamma_n$ (which is a 'lift' of the inclusion $\gamma_m \hookrightarrow \gamma_n$) affording a *canonical* filtration of $\Gamma_n(G)$ such that the associated graded object is an explicitly calculable Lie algebra. This Lie algebra comes with a natural surjection onto the Lazard Lie algebra. If G is a nilpotent group then the canonical filtration of $\Gamma_n(G)$ is finite; since the successive quotients are quotients of things known to us we get estimates for the order of $\Gamma_n(G)$ when G is finite nilpotent and for the rank of $\Gamma_n(G)$ when G is nilpotent but not finite. When $n = 2$ the exact sequence

$$0 \to H_2(G, \mathbf{Z}) \to \Gamma_2(G) \to \gamma_2(G) \to 1$$

leads to estimates for $H_2(G, \mathbf{Z})$ which seem quite different from the existing estimates. The point here is that our filtration of Γ_2, and hence of H_2, does not come from a choice of subgroup (as in the 'homological' approach) but from the images in Γ_2 of the higher functors $\Gamma_3, \Gamma_4, \ldots$ and the results are, accordingly, of a new nature (see §10).

Our definition of Γ_n is as follows. We define n-central extensions and projective n-central extensions. Using the commutator calculus, we show that if $U \to G$ is a projective n-central extension then $\gamma_n(U)$ is *functorial* in G. Then $\Gamma_n(G) = \gamma_n(U)$. It is a *canonical* central extension of $\gamma_n(G)$. If one uses a free n-central extension, one gets Baer's formula for $\Gamma_n(G)$. This already shows that the torsion of the kernel of a free n-central extension is an abelian group which is an invariant of G (see §5). This simple proof should be compared with the proof in [11] of the invariance of the torsion in the free central case. In §6 is proved the finiteness of $\Gamma_n(G)$ when G is finite. It is shown to follow from Schur's original result that G' is finite if G has a central subgroup of finite index. In §7 the canonical filtration is introduced and its finiteness in the nilpotent case and its triviality in the perfect case are proved. The Lie structure on the associated graded group $\operatorname{gr}\Gamma(G)$ is introduced in §8 and the main theorem, which is a formula for $\operatorname{gr}\Gamma(G)$, is proved in §9. The result is that $\operatorname{gr}\Gamma(G)$ is the 'free' Lie algebra on the abelianization of G. Finally, §10 gives the application of this theorem to estimates of order and rank of $\Gamma_n(G)$. These estimates seem to raise interesting questions on the orders of the homogeneous components of the free Lie algebra (over $\mathbf{Z}$) modulo relations which render its first component finite.

In this paper I use a slightly different indexing system than the usual. The definition of n-central extension is such that a 1-central extension is an isomorphism while old fashioned central extensions are '2-central'. Similarly, the higher centers are defined by $Z_1(G) = \{1\}$, $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$, where $Z(\ )$ denotes 'center'. My reasons for doing this are twofold. The first is that this is the correct indexing relative to the weight of commutators; for example, an extension is n-central (in this indexing) iff certain weight n commutators are trivial. The second reason is that this indexing gives the 'correct' graded structure for the Lie algebra $\operatorname{gr}\Gamma(G)$. By 'correct grading' I mean that the Lie algebra starts at degree 1, i.e., vanishes in degrees ≤ 0 and is generated by the elements of degree 1. This is very convenient.

The referee has brought to my attention the works [3], [8] and [10], which treat the more general, varietal, definition of the Baer invariants and also give applications of these generalized invariants to isologism theory. These works have an overlap with sections 1–4, 6 and the part of section 7 dealing with perfect groups of this paper. But it should be added that these works do not treat the canonical filtration and its associated Lie algebra.

1. Higher central extensions

A central extension of a group G is a homomorphism onto G whose kernel is central. In our somewhat unorthodox numbering, such extensions are called 2-central, a 1-central extension being simply an isomorphism. The reasons for this numbering are given in the introduction.

(1.1) Definition. An n-central extension ($n \geq 2$) is a surjective homomorphism $X \to G$ whose kernel contains a subgroup A which is central in X such that the induced map $X/A \to G$ is an $(n-1)$-central extension. In other words, an n-central extension is one whose kernel is contained in the n-th center of X.

Thus, if $n \geq m$, an m-central extension is also n-central. An extension is *strictly* n-central if it is n and not $(n-1)$-central.

Clearly, an n-central extension of the trivial group is a nilpotent group of nilpotence class $< n$. Thus higher central extensions can be thought of as a 'relativization' of nilpotent groups.

It is clear how to define morphisms between n-central extensions: these are just commutative diagrams

$$\begin{array}{ccc} X & \to & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \to & H \end{array}$$

of homomorphisms where $X \to G$ and $Y \to H$ are n-central extensions. The map g is also called a *lift* of f. Note that, given f, it may have no lift and, even when a lift does exist, it may not be unique.

We shall denote by $[a_1, \ldots, a_n]$ the higher commutator $[[a_1, \ldots, a_{n-1}], a_n]$. Similarly $[A_1, \ldots, A_n]$, where $A_1, \ldots, A_n$ are normal subgroups, is defined. The notation $[A, mB]$ is explained above.

(1.2) LEMMA. *Let $\varphi : X \to G$ be a group extension with kernel N. Then it is n-central iff the higher commutator subgroup $[N, (n-1)X]$ is trivial. In particular, N is nilpotent in this case.*

PROOF. For $n = 1,2$ this is trivial. In general we use induction. Let A be a central subgroup, contained in N, such that the associated map $X/A \to G$ is $(n-1)$-central. Denoting X/A by $\bar{X}$, N/A by $\bar{N}$, etc., we know by the inductive assumption (now $n > 2$) that $[\bar{N}, (n-2)\bar{X}] = 1$, which implies that $[N, (n-2)X] \subseteq A$. This means that

$$[N, (n-1)X] = [[N, (n-2)X], X] \subseteq [A, X] = \{1\}.$$

Conversely, if $[N, (n-1)X] = \{1\}$ then $A = [N, (n-2)X]$ is central in X and $[\bar{N}, (n-2)\bar{X}] = 1$, where the bar denotes 'modulo A' again. So $\bar{\varphi} : \bar{X} \to G$ is an $(n-1)$-central extension by the inductive assumption. ■

(1.3) COROLLARY. *Let $X \to G$ be an extension with kernel N. Then the associated extension $X/[N, (n-1)X] \to G$ is n-central.*

The corollary will be used primarily when X is a free group.

2. Commutator calculus

We shall use the 'Jacobi–Hall' identity in its following simple form: if A, B, C are normal subgroups of a group then

$$[[A,B],C] \subseteq [[A,C],B] \cdot [A,[B,C]].$$

Instead of $[[A,B],C]$ we can write $[A,B,C]$ as mentioned before.

(2.1) LEMMA. *Let G be a group and H a normal subgroup. Then, for every $n \geq 0$, $m \geq 1$, $[[H, nG], \gamma_m(G)] \subseteq [H, (n+m)G]$. In particular, $[H, \gamma_n(G)] \subseteq [H, nG]$.*

PROOF. Induction on m. If $m = 1$, this is true for every n by definition. Assuming the result for a given m (and every n), it follows for $m + 1$ (and every n) by Jacobi:

$$\begin{aligned}[[H,nG],\gamma_{m+1}(G)] &= [[H,nG],[\gamma_m(G),G]] \\ &\subseteq [[[H,nG],\gamma_m(G)],G]\cdot[\gamma_m(G),[[H,nG],G]] \\ &\subseteq [H,(n+m+1)G]. \end{aligned}$$ ∎

This lemma implies a more general result. To state it, let us define a weight m higher commutator in $x_1,\ldots,x_r$ to be some x_i, if $m = 1$, and $[u,v]$ where u,v are commutators (in the same letters) of lesser weight and of weight sum m, if $m > 1$.

(2.2) PROPOSITION (G,H as above). *Every commutator of weight n in $x_1,\ldots,x_n$ for which some $x_j \in H$ is in $[H,(n-1)G]$.*

PROOF. If $n = 1$, this is clear. If $n > 1$ and the commutator is $[u,v]$, and x_j appears in u, say, then by the inductive hypothesis $u \in [H,(r-1)G]$ and $v \in \gamma_{n-r}(G)$ for some r satisfying $1 \le r < n$. The lemma now gives $[[H,(r-1)G], \gamma_{n-r}(G)] \subseteq [H,(n-1)G]$. ∎

(2.3) COROLLARY. *Let $X \to G$ be an extension with kernel N. Then it is n-central iff every weight n commutator in $x_1,\ldots,x_n \in X$, of which at least one $x_j \in N$, is* 1.

PROOF. This follows directly from (2.2) and (1.2). ∎

This is the first reason for my calling an extension n-central, as I did in (1.1). Another motivation for this unconventional numeration is mentioned in the introduction.

The following is a key technical result.

(2.4) THEOREM. *Let $\varphi: X \to G$ be an n-central extension; then, for every n elements $x_1,\ldots,x_n$ of X, the commutator of weight n $[x_1,\ldots,x_n]$ only depends on $\varphi(x_1),\ldots,\varphi(x_n)$.*

In other words, if $a_1,\ldots,a_n \in \ker(\varphi)$ then $[a_1x_1,\ldots,a_nx_n] = [x_1,\ldots,x_n]$.

PROOF. Let $A \subseteq \ker(\varphi)$ be a central subgroup such that the extension $\bar{\varphi}: \bar{X} = X/A \to G$ is $(n-1)$-central. By induction we know that $[\bar{x}_1,\ldots,\bar{x}_{n-1}] \in \bar{X}$ depends only on $\bar{\varphi}(\bar{x}_i) = \varphi(x_i) \in G$ $(i = 1,\ldots,n-1)$ so that $[a_1x_1,\ldots,a_{n-1}x_{n-1}] = b\cdot[x_1,\ldots,x_{n-1}]$ where $b \in A$ is a certain function of $a_1,\ldots,a_{n-1}$. As b is central, $[b\cdot[x_1,\ldots,x_{n-1}],a_nx_n] = [[x_1,\ldots,x_{n-1}],a_nx_n]$ and it only remains to show that $[[x_1,\ldots,x_{n-1}],a_nx_n] = [[x_1,\ldots,x_{n-1}],x_n]$.

We will use the identity

$$(*) \qquad [x,yz] = [x,y]\cdot[y,[x,z]]\cdot[x,z],$$

which is easily checked. In this identity let $x = [x_1,\dots,x_{n-1}]$, $y = a_n$, $z = x_n$. Then by (2.1) and (1.2), $[[x_1,\dots,x_{n-1}],a_n] = 1$ and also $[a_n,[x_1,\dots,x_n]] = 1$. So we see that $[[x_1,\dots,x_{n-1}],a_nx_n] = [x_1,\dots,x_n]$, as required. ■

A reformulation of (2.4) which is free of 'n-central extensions' is this: if X is a group and $x_1,\dots,x_n \in X$ then the higher commutator $[x_1,\dots,x_n]$ only depends on the classes of the x_i's in $X/Z_n(X)$.

The following two corollaries of (2.4) will be important.

(2.5) PROPOSITION. *Let $X \to G$, $Y \to H$ be n-central extensions, $f: G \to H$ a homomorphism. If $g_1, g_2: X \to Y$ are liftings of f then $\gamma_n(g_1) = \gamma_n(g_2)$.*

In other words, for every $x_1,\dots,x_n \in X$,

$$[g_1(x_1),\dots,g_1(x_n)] = [g_2(x_1),\dots,g_2(x_n)].$$

PROOF. This follows directly from (2.4) because, for every $x \in X$, $g_1(x)$ and $g_2(x)$ differ by an element of $\ker(Y \to H)$. ■

(2.6) PROPOSITION. *Let $f: G \to H$ be a surjective group homomorphism, $X \to G$ and $Y \to H$ n-central extensions and $g: X \to Y$ a lifting of f. Then $\gamma_n(g): \gamma_n(X) \to \gamma_n(Y)$ is onto.*

PROOF. Since commutators of the type $[y_1,\dots,y_n]$ generate $\gamma_n(Y)$, it would suffice to show that they are in the image of $\gamma_n(g)$. The commutativity of the diagram

$$\begin{array}{ccc} X & \to & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \to & H \end{array}$$

and the surjectivity of f imply that for each $i = 1,\dots,n$ there is $u_i \in X$ whose image in H is the same as that of y_i. By (2.4)

$$[y_1,\dots,y_n] = [g(u_1),\dots,g(u_n)] = g([u_1,\dots,u_n]) \in g(\gamma_n(X)). \qquad ■$$

3. Projective extensions

(3.1) DEFINITION. An n-central extension $\varphi: U \to G$ is projective if for every n-central extension $\psi: V \to H$ and homomorphism $f: G \to H$ there is a lift $g: U \to V$, i.e. such that $\psi \circ g = f \circ \varphi$.

Given a group G, let $\pi : F \to G$ be an epimorphism with F a free group. If $R = \ker(\pi)$ then the extension $\bar{\pi} : F/[R,(n-1)F] \to G$ is n-central by (1.3). Such extensions will be called *free*. Obviously, every group has many free n-central extensions.

(3.2) LEMMA. *Free extensions are projective.*

PROOF. Let $f : G \to H$ be a homomorphism and $V \to H$ an n-central extension. We need to construct a map $g : F/[R,(n-1)F] \to V$ lifting f. As F is free, a map $h : F \to V$ lifting f exists. So $h(R) \subseteq \ker(V \to H)$ and it follows from (1.2) that $h([R,(n-1)F]) = 1$. Thus h gives rise to a map $\bar{h} = g : F/[R,(n-1)F] \to V$, satisfying our requirements. ∎

There is another definition of 'freeness' which rests on the concept of a free basis: in this definition, an n-central extension $X \to G$ is free on a basis $x_1, \ldots, x_m$ if for every n-central extension $Y \to H$, homomorphism $f : G \to H$ and elements $y_1, \ldots, y_m \in Y$, such that y_i and x_i have the same image in H for every i, there is a unique lifting $g : X \to Y$ satisfying $g(x_1) = y_1, \ldots, g(x_m) = y_m$. It is not hard to show that the two definitions of freeness are the same, and we leave this to the reader.

From the existence of projective extensions we have

(3.3) LEMMA. *Let $\varphi : U \to G$ be an n-central extension. Then it is projective iff for every n-central extension $\psi : V \to G$ there exists a morphism $\tau : U \to V$ such that $\varphi = \psi \circ \tau$ (i.e., a lifting of* id_G*).*

PROOF. The 'only if' part is trivial. Conversely, we must exhibit for every $f : G \to H$ and n-central extension $Y \to H$ a lift $g : U \to Y$ for f. Let $W \to G$ be a projective n-central extension. Then $\tau : U \to W$ lifting id_G exists by assumption and $\rho : W \to Y$ lifting f exists by projectiveness. Then $\rho \circ \tau$ is a lift of f. ∎

(3.4) *Examples of non-free projective extensions.* The examples that follow seem quite 'artificial,' and it would be interesting to construct more natural ones. They are based on the fact, proved below in §5, that the torsion part of the kernel of a free n-central extension is an invariant of the group and is, in fact, *commutative.* If $U \to G$ is a free n-central extension and P is a nilpotent group of nilpotence class $< n$, then it is easy to see that the extension $U \times P \to G$ (where $1 \times P$ goes to 1) is n-central and projective. Clearly, P can be so chosen that $\mathrm{torsion}(\ker(U \to G))$ is not isomorphic to $\mathrm{torsion}(\ker(U \times P \to G))$. This gives the desired examples.

Let $1 \to R \to F \to G \to 1$ be a free presentation of G. If $m \leq n$, then $[R,(n-1)F] \subseteq [R,(m-1)F]$ and there is a commutative diagram

$$\begin{array}{ccc} F/[R,(n-1)F] & \searrow & \\ \downarrow & & G. \\ F/[R,(m-1)F] & \nearrow & \end{array}$$

If we denote $F/[R,(n-1)F]$ by U and $R/[R,(n-1)F]$ by N, it is clear that $U/[N,(m-1)U] = F/[R,(m-1)F]$. This property of free extensions is shared by projective extensions.

(3.5) Lemma. *Let $\varphi: U \to G$ be a projective n-central extension with kernel N. Then, for $m \leq n$, $\bar{\varphi}: U/[N,(m-1)U] \to G$ is a projective m-central extension.*

Proof. Let us denote $U/[N,(m-1)U]$ by V. We must prove that if $X \to H$ is an m-central extension and $f: G \to H$ a homomorphism that there exists $g: V \to X$ lifting f. Thinking of $X \to H$ as an n-central extension (which it is because $n \geq m$), there is a map $h: U \to X$ lifting f. So h maps N into $\ker(X \to H) = L$. By (1.2) $[L,(m-1)X] = \{1\}$, so it follows that $h([N,(m-1)U]) = \{1\}$ and h defines a map $\bar{h} = g: U/[N,(m-1)U] = V \to X$ which lifts f, as required. ∎

4. The functors Γ_n

In homological algebra, the standard procedure to define derived functors of the functor Φ is this. Given an object M (in an abelian category with enough projectives) take a projective resolution of it $\cdots \to P_1 \to P_0 \to M \to 0$, apply Φ to get a complex $\cdots \to \Phi(P_1) \to \Phi(P_0) \to 0$ and compute the homology of this complex. We do something simpler.

(4.1) Definition. Let $U \to G$ be a projective n-central extension. We define $\Gamma_n(G)$ to be $\gamma_n(U)$. If $f: G \to H$ is a homomorphism, we define $\Gamma_n(f)$ as follows. Let $V \to H$ be a projective n-central extension, and $\tilde{f}: U \to V$ be a lift of f. Then $\Gamma_n(f)$ is $\gamma_n(\tilde{f})$, i.e., it is $\tilde{f}$ considered as a map from $\gamma_n(U)$ to $\gamma_n(V)$.

Proof that Γ_n is a functor. We have to exhibit, given any two ways to compute $\Gamma_n(G)$, a *unique* isomorphism between them so that, for any *three* ways to compute $\Gamma_n(G)$, the composition of any two of these isomorphisms would equal the third. So let $\varphi: U \to G$ and $\psi: W \to G$ be two projective n-central extensions and $h: U \to W$ such that $\psi \circ h = \varphi$. From (2.4) it follows that $\gamma_n(h): \gamma_n(U) \to \gamma_n(W)$ is unique, i.e. is independent of h. This implies that $\gamma_n(h)$ is the unique isomorphism required since if $h': W \to U$ is such that $\varphi \circ h' = \psi$ then $\gamma_n(h \circ h') = \gamma_n(h) \circ \gamma_n(h')$ must be the identity of $\gamma_n(W)$, being the (unique!) restriction to $\gamma_n(W)$ of the lift to W of id_G. Similarly, $\gamma_n(h' \circ h) = \mathrm{id}_{\gamma_n(U)}$.

The proof that Γ_n of a morphism is well defined and satisfies $\Gamma_n(1) = 1$ and $\Gamma_n(f \circ g) = \Gamma_n(f) \circ \Gamma_n(g)$ is similar and as easy, and is left to the reader. ■

If $\varphi : U \to G$ is a projective n-central extension, then clearly φ maps $\gamma_n(U)$ onto $\gamma_n(G)$, so $\gamma_n(U)$ is an extension of $\gamma_n(G)$. If $N = \ker(\varphi)$ then $\ker(\gamma_n(\varphi)) = N \cap \gamma_n(U)$. But, by (1.2), $[N, \gamma_n(U)] = 1$, implying that $N \cap \gamma_n(U)$ is *central* in $\gamma_n(U)$. Now if $\psi : W \to G$ is another projective n-central extension and $f : U \to W$ lifts id_G, then clearly $\gamma_n(\psi) \circ \gamma_n(f) = \gamma_n(\varphi)$. This proves

(4.2) Proposition. *$\Gamma_n(G)$ is a canonical central extension of $\gamma_n(G)$, i.e., there is a morphism of functors $\Gamma_n \to \gamma_n$ which makes $\Gamma_n(G) \to \gamma_n(G)$ a central extension.*

In order to see more concretely what Γ_n is, we look at free extensions. Let $\pi : F \to G$ be a surjection, with F free, and $R = \ker(\pi)$. Let $F/[R,(n-1)F] \to G$ be the associated n-central extension, then

$$\Gamma_n(G) = \gamma_n(F/[R,(n-1)F]) = \gamma_n(F)/[R,(n-1)F]. \tag{4.3}$$

As mentioned in the introduction, the invariance of these groups was first shown by Baer [1], who considered them as invariants of presentations.

(4.4) Examples. (1) Suppose G is a free group. Then the identity is an n-central extension for every n, so $\Gamma_n(G) = \gamma_n(G)$ in this case. Note that if we compute $\Gamma_n(G)$ with other free presentations then the isomorphism $\Gamma_n(G) \stackrel{\approx}{\to} \gamma_n(G)$ becomes less obvious!

(2) Suppose G is a free abelian group, say $G \approx \mathbf{Z}^m$. Let F be a free group of rank m and $\pi : F \to G$ a surjection mapping a basis of F to a basis of G, so that $R = \ker(\pi)$ is $[F,F]$. Thus $[R,(n-1)F] = \gamma_{n+1}(F)$ and $\Gamma_n(G) = \gamma_n(F)/\gamma_{n+1}(F)$. Note that these are the components of the Lie algebra associated with the lower central series of F.

(3) Although we shall see below the precise structure of $\Gamma_n(G)$ when G is a *finite abelian* group, we now show some properties of it using simple commutator identities. Say G is a product of r cyclic groups of order $m_1, \ldots, m_r$. Let F be a free group on r generators $x_1, \ldots, x_r$ and $\pi : F \to G$ the map sending these generators to the respective generators of the cyclic groups. Then $R = \ker(\pi)$ contains the commutator subgroup $[F,F]$ so that $\gamma_n(F)/[R,(n-1)F]$ is a quotient of the abelian group $\gamma_n(F)/\gamma_{n+1}(F)$. If $a \in \gamma_{n-1}(F)$ then, by (2.1), $[a, x_i^{m_i}] \in [R,(n-1)F]$. As $F/[R,(n-1)F]$ is a quotient of $F/\gamma_{n+1}(F)$, it is nilpotent of class $\leq n$ and commutators of weight n in it are central. It follows that in

$F/[R,(n-1)F]$, the commutator $[\bar{a},\bar{x}_i]$ is central and hence it commutes with $\bar{a}$ and $\bar{x}_i$. Now we use a well-known commutator identity (see [14], p. 92):

$$[x,y^m] = [x,y]^m \qquad \text{if } [x,y] \text{ commutes with } x \text{ and } y.$$

But, in $\gamma_n(F)/[R,(n-1)F] \subset F/[R,(n-1)F]$, $[\bar{a},\bar{x}_i^{m_i}] = 1$ so $[\bar{a},\bar{x}_i]^{m_i} = 1$ and we see that the abelian group $\gamma_n(F)/[R,(n-1)F] = \Gamma_n(G)$ is a torsion group and, in fact, an exponent of G is an exponent for it.

5. Application: Invariance of the torsion of the kernel of a free extension

In [11], I proved that the torsion of the kernel of a free central extension is an invariant of the group. The idea was to show that changing a presentation by an 'elementary Tietze move' does not change this group. However, it seems to be difficult to analyze the effect of a Tietze move on a free n-central extension for $n > 2$, so that the proof of [11] does not generalize to give the invariance of the torsion of the kernel of a free n-central extension for $n > 2$. Thus it comes as a pleasant bonus to find that, in fact, this group (it *is* a group, since the kernel of an n-central extension is nilpotent; see (1.2)) is the torsion part of $\ker(\Gamma_n \to \gamma_n)$.

(5.1) THEOREM. *Let $F/[R,(n-1)F] \to G$ be a free n-central extension. Then the torsion of its kernel is equal to the torsion of $(R \cap \gamma_n(F))/[R,(n-1)F]$ and is therefore central in $\gamma_n(F)/[R,(n-1)F]$ and equal to* torsion$(\ker(\Gamma_n(G) \to \gamma_n(G)))$ *(a manifest invariant).*

We denote the group torsion $(R/[R,(n-1)F])$ by $\tau_n(G)$.

PROOF. Let $x \in R$ be such that its class $\bar{x} \in R/[R,(n-1)F]$ is of finite order, say $\bar{x}^m = 1$ so that $x^m \in [R,(n-1)F]$. In particular, $x^m \in \gamma_n(F)$. But as $F/\gamma_n(F)$ is torsion free, $x \in \gamma_n(F)$. So $x \in R \cap \gamma_n(F)$ and $\bar{x} \in R \cap \gamma_n(F)/[R,(n-1)F] = \ker(\Gamma_n(G) \to \gamma_n(G))$, and it is shown in (4.2) that $\Gamma_n(G) \to \gamma_n(G)$ is a central extension. ∎

Because it is important, and for later applications, we single out the case $n = 2$.

(5.2) COROLLARY. $\tau_2(G) = $ torsion$(H_2(G,\mathbf{Z}))$. *In particular, if G is finite, $\tau_2(G) = H_2(G,\mathbf{Z})$.*

PROOF. Let $F/[R,F] \to G$ be a free central extension. Then $\tau_2(G) =$ torsion$(R/[R,F])$ equals, by (5.1), torsion$(R \cap [F,F]/[R,F])$. But $R \cap [F,F]/[R,F]$ is $H_2(G,\mathbf{Z})$ by 'Hopf's formula'. ∎

What is the nature of the higher torsion functors τ_n $(n \geq 3)$? Are they also connected with the homology of G? This seems an interesting problem. Even without an explicit formula for τ_n in homological terms, the following would be useful. If G is a finite group, G_p a p-Sylow subgroup of it, then it is shown below (see (10.3)) that $\tau_n(G_p)$ is p-primary. So the inclusion $G_p \subset G$ induces $\tau_n(G_p) \to \tau_n(G)(p)$ where $A(p) = \{a \in A : a \text{ of order a power of } p\}$. Is this map surjective, as in homology? This would follow if the functor τ_n had a transfer map (which it does for $n = 2$, by virtue of (5.2)).

6. Finiteness

In 1952, Baer proved [2] that if the n-th center of a group X has finite index in X then $\gamma_n(X)$ is finite. This generalized Schur's result, which is the case $n = 2$. (Recall that, for us, $Z_1 = \{1\}$, $Z_{n+1}(X)/Z_n(X) = Z(X/Z_n(X))$.) The machinery of the present paper applies to Baer's situation as follows. Denote $X/Z_n(X) = G$. Then X is an n-central extension of G. Let $U \to G$ be a projective n-central extension of G. There is a morphism $f: U \to X$ lifting id_G and this morphism maps $\gamma_n(U) = \Gamma_n(G)$ *onto* $\gamma_n(X)$ by (2.6). Thus the following lemma is a rephrasing of a special case of (2.6).

(6.1) Lemma. *If $X \to G$ is an n-central extension of G then $\gamma_n(X)$ is a quotient of $\Gamma_n(G)$.* ∎

Baer's theorem follows now from

(6.2) Theorem. *If G is a finite group then $\Gamma_n(G)$ is finite.*

But (6.2) gives more, in fact. Combined with (6.1), it shows that $|\Gamma_n(G)|$ is both the maximum and the least common multiple of the numbers $|\gamma_n(X)|$ as X runs on the set of n-central extensions of the given group G. This underscores the importance of having good estimates for the orders of the groups $\Gamma_n(G)$. Very good estimates are given in §10 for nilpotent groups and a positive answer to the question at the end of §5 (is $\tau_n(G_p) \to \tau_n(G)(p)$ onto?) would give the correct estimate for $|\Gamma_n(G)|$ in general.

To prove (6.2), we will need three auxiliary results. Essentially, these will reduce the proof to Schur's original theorem.

(6.3) Lemma. *Let B be a torsion free abelian group, A a subgroup of finite index. Then an automorphism of B fixing A elementwise is the identity.*

PROOF. The automorphism extends to $\mathbf{Q} \otimes_{\mathbf{Z}} B = B_{\mathbf{Q}}$ and is the identity on $A_{\mathbf{Q}}$. But $A_{\mathbf{Q}} = B_{\mathbf{Q}}$. As $B \subset B_{\mathbf{Q}}$ the automorphism is, necessarily, the identity. ∎

(6.4) LEMMA. *Let G be a finitely generated group, H a normal subgroup which is normally finitely generated. Then $[G,H]$ is normally finitely generated.*

PROOF. $[G,H]$ is the smallest normal subgroup such that division by it 'makes H central'. But if $x_1,\dots,x_r$ generate H normally and $y_1,\dots,y_s$ generate G, let L be the normal subgroup generated by $\{[x_i,y_j] : 1 \le i \le r,\ 1 \le j \le s\}$. Then $L \subseteq [G,H]$ and, in G/L, $\bar{x}_i$ are central so that H/L is central in G/L. It follows that $L = [G,H]$, so $[G,H]$ is finitely generated normally by $\{[x_i,y_j]\}$. ∎

(6.5) PROPOSITION. *Suppose A,B are normal subgroups of a group G, $A \subseteq B$ and $(B:A) < \infty$. If B is finitely generated modulo $[A,G]$ then $([B,G]:[A,G]) < \infty$.*

For example, if A (and thus B) is normally finitely generated then B is finitely generated modulo $[A,B]$, i.e., $B/[A,G]$ is finitely generated.

PROOF. Dividing out $[A,G]$, it remains to show that $[B,G]$ is finite if A is central (Schur's theorem is the case $B = G$). By Schur, $[B,B]$ is finite, so it suffices to show that $[B,G]$ is finite modulo $[B,B]$. Thus we can assume that B is abelian (and finitely generated). The torsion of B, say T, is now finite and normal in G, so it would suffice to show that $[B,G]$ is finite modulo T. Thus we can assume that B is also torsion free. We will show that these assumptions make B central, i.e., $[B,G] = 1$: if $x \in G$ then conjugation by it fixes A as A is central so, by (6.3), it is the identity on B, i.e., B is central in G. ∎

(6.6) PROOF OF (6.2). Let $\varphi: U \to G$ be a projective n-central extension with U finitely generated, and let $N = \ker(\varphi)$. As $(U:N) < \infty$, N is finitely generated and, by (6.4), $[N,kU]$ is normally finitely generated for every k. Applying (6.5) $(n-1)$ times, we see that $(\gamma_n(U):[N,(n-1)U]) < \infty$. But, by (1.2), $[N,(n-1)U] = 1$! ∎

7. The canonical filtration of Γ_n

(7.1) If $\varphi: U \to G$ is an n-central extension, $x_1,\dots,x_n \in G$ and $\varphi(u_i) = x_i$ for $i = 1,\dots,n$ then, by (2.4), the commutator $[u_1,\dots,u_n]$ depends only on $x_1,\dots,x_n$. We denote it by

$$c_U(x_1,\dots,x_n).$$

If $V \to H$ is another n-central extension, $f: G \to H$ a homomorphism and $\tilde{f}: U \to V$ a lifting of f then it is easily seen that the function c_U has the following *functorial* behaviour:

$$\tilde{f}c_U(x_1, \dots, x_n) = c_V(f(x_1), \dots, f(x_n)).$$

In particular, if $U \to G$ is a projective n-central extension then $c_U(x_1, \dots, x_n) \in \Gamma_n(G)$ defines a *canonical* element of $\Gamma_n(G)$. We denote it $c(x_1, \dots, x_n)$.

What are the properties of this function? As seen in (4.4), $c(x_1, \dots, x_n)$ is, in some cases, simply the commutator $[x_1, \dots, x_n]$, which is a pretty complicated function. We know that in order to simplify the higher commutator, for example to 'linearize' it, one sometimes looks at its residue class in $\gamma_n(G)/\gamma_{n+1}(G) = \mathrm{gr}_n \gamma(G)$. In our case, we will need to have a natural morphism from $\Gamma_{n+1}(G)$ to $\Gamma_n(G)$.

If $U \to G$ is an m-central extension then it is also n-central for every $n \geq m$. So if $V \to G$ is a projective n-central extension there is a map $f: V \to U$ such that the diagram

$$\begin{array}{ccc} V & \searrow & \\ {\scriptstyle f}\downarrow & & G \\ U & \nearrow & \end{array}$$

commutes. By (2.5), $\gamma_n(f)$ is unique, i.e. independent of f. Moreover, from (2.6) we know that the image of $\gamma_n(f)$, which is $f(\gamma_n(V))$, is equal to $\gamma_n(U)$. Taking $U \to G$ to be a projective m-central extension, we get a map $f: \gamma_n(V) = \Gamma_n(G) \to \Gamma_m(G) = \gamma_m(U)$. We claim that, in fact, this map is canonical:

(7.2) PROPOSITION. *There is a morphism of functors* $\Gamma_n \xrightarrow{\Phi^n_m} \Gamma_m$ $(n \geq m)$ *which, in the notation above, on* $\gamma_n(V)$ *is* f.

PROOF. We must show that if $U_1 \to G$ is m-central projective, $V_1 \to G$ is n-central projective, $f_1: V_1 \to U_1$ lifts id_G and $g_V: V_1 \to V$ and $g_U: U_1 \to U$ also lift id_G, then the diagram

$$\begin{array}{ccc} \gamma_n(V_1) & \xrightarrow{g_V} & \gamma_n(V) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ \gamma_m(U_1) & \xrightarrow[g_U]{} & \gamma_m(U) \end{array}$$

commutes. This follows directly from (2.5) or by noting that both $f \circ g_V$ and $g_U \circ f_1$ send $c_{V_1}(x_1, \dots, x_n)$ to $c_U(x_1, \dots, x_n)$. ■

It is easily seen that the maps Φ_m^n have the transitivity property

$$\phi_l^m \circ \Phi_m^n = \Phi_l^n \qquad \text{if } n \geq m \geq l.$$

Thus we get a *canonical* filtration of $\Gamma_n(G)$ by the images in it of $\Gamma_{n+1}(G)$, $\Gamma_{n+2}(G), \ldots, \Gamma_n(G) \supseteq \operatorname{im}\Gamma_{n+1}(G) \supseteq \operatorname{im}\Gamma_{n+2}(G) \supseteq \ldots$.

(7.3) LEMMA. *This filtration (henceforth 'the canonical filtration') is central, i.e., each* $\operatorname{im}\Gamma_{n+k}$ *is normal in* Γ_n *and* $\operatorname{im}\Gamma_{n+k}/\operatorname{im}\Gamma_{n+k+1}$ *is central in* $\Gamma_n/\operatorname{im}\Gamma_{n+k+1}$.

PROOF. If $U \to G$ is a projective n-central extension then $\Gamma_n(G) = \gamma_n(U)$ and $\operatorname{im}\Gamma_{n+k}(G)$ is equal to $\gamma_{n+k}(U)$, as seen above. Hence the lemma is clear. ∎

Thus the quotient

$$\Gamma_n(G)/\operatorname{im}\Gamma_{n+1}(G) = \operatorname{gr}_n \Gamma(G)$$

is an abelian group and will be considered as an additive group.

What kind of filtration is the canonical filtration? It will be useful to have some

(7.4) EXAMPLES. (1) If G is a *free* group we know that $\Gamma_n(G) = \gamma_n(G)$ and, clearly, Φ_m^n is the inclusion $\gamma_n(G) \subseteq \gamma_m(G)$ and $\operatorname{gr}_n \Gamma(G) = \operatorname{gr}_n \gamma(G)$. This example will be used later to deduce properties of $\operatorname{gr}\Gamma$ from those of $\operatorname{gr}\gamma$.

(2) If G is an *abelian* group then the map $\Phi_m^n : \Gamma_n(G) \to \Gamma_m(G)$ is the trivial map (with image $\{1\}$) if $n > m$. (In particular, $\Gamma_m(G)$ is abelian.) Indeed, let $U \to G$ be a projective m-central extension, with kernel N. Then $N \supseteq [U, U]$ as G is abelian and thus $[N, (m-1)U] \supseteq [[U,U],(m-1)U] = \gamma_{m+1}(U)$. But, by (1.2), $[N,(m-1)U] = 1$, so $\gamma_{m+1}(U) = \{1\}$. As $\Gamma_m(G) = \gamma_m(U)$ and $\operatorname{im}\Gamma_n(G) = \gamma_n(U)$, we see that $\operatorname{im}\Gamma_n(G) = \{1\}$ for $n > m$.

This example has the following, extremely useful, generalization.

(7.5) PROPOSITION. *If G is nilpotent of class c then for each $n \geq 1$ the canonical filtration of* $\Gamma_n(G)$ *descends to* $\{1\}$ *in c steps, i.e.,* $\operatorname{im}\Gamma_{n+c}(G) = \{1\}$. *In particular,* $\Gamma_n(G)$ *is nilpotent of class c.*

As $\Gamma_n(G) \to \gamma_n(G)$ is a central extension, $\Gamma_n(G)$ is, of course, abelian if $n > c$. But even then, the canonical filtration may still be nontrivial.

PROOF. The same as for G abelian. Let $U \to G$ be a projective n-central extension with kernel N. As $\gamma_{c+1}(G) = \{1\}$, $N \supseteq \gamma_{c+1}(U)$ and $[N,(n-1)U] \supseteq \gamma_{n+c}(U)$. But, from (1.2), $[N,(n-1)U] = \{1\}$ so $\gamma_{n+c}(U) = \operatorname{im}\Gamma_{n+c}(G) = \{1\}$. ∎

At the other extreme is the case that G is perfect, i.e., $G = [G,G]$. In this case we will show that $\Gamma_n(G) \to \gamma_n(G) = G$ is the universal central extension and each Φ^n_m is an isomorphism for $n \geq m \geq 2$, so that the canonical filtration is trivial.

We will need the following important property:

(7.6) PROPOSITION. *Let $U \to G$ be an n-central extension and let $\bar{c}_U(x_1,\ldots,x_n)$ be the residue class in $\gamma_n(U)/\gamma_{n+1}(U)$ of $c_U(x_1,\ldots,x_n)$. Then $\bar{c}_U$ is additive, i.e., for each i,*

$$\bar{c}_U(x_1,\ldots,x_ix_i',\ldots,x_n) = \bar{c}_U(x_1,\ldots,x_i,\ldots,x_n) + \bar{c}_U(x_1,\ldots,x_i',\ldots,x_n).$$

In particular, if U is projective then $c_U(x_1,\ldots,x_n)$ was denoted by $c(x_1,\ldots,x_n)$ and its residue class in $\gamma_n(U)/\gamma_{n+1}(U) = \mathrm{gr}_n\,\Gamma(G)$ will be denoted $\{x_1,\ldots,x_n\}$. Then the proposition says that the function $\{x_1,\ldots,x_n\}$ is 'multilinear'.

PROOF. This can be proved directly, but we can use example (1) above to 'reduce' the proof to 'known' facts. If F is a free group then $\Gamma_n(F) = \gamma_n(F)$ and, for $y_1,\ldots,y_n \in F$, $\{y_1,\ldots,y_n\}$ is simply the class in $\gamma_n(F)/\gamma_{n+1}(F)$ of the commutator $[y_1,\ldots,y_n]$. For γ_n/γ_{n+1}, one knows that $\{y_1,\ldots,y_n\}$ is just the Lie product of the classes $\{y_i\} \in F/[F,F] = \mathrm{gr}_1\,\gamma(F)$, so it is clear that $\{y_1,\ldots,y_n\}$ is a multilinear function of its variables. If F is a free group on letters $y_1,\ldots,y_i,y_i'$, $\ldots,y_n$, there is a homomorphism $F \overset{f}{\to} G$ sending y_j to x_j and y_i' to x_i'. This map can be lifted to a map $F \overset{g}{\to} U$ and it is easily seen that the induced map from $\mathrm{gr}_n\,\Gamma(F)$ to $\gamma_n(U)/\gamma_{n+1}(U)$ takes $\{y_1,\ldots,y_iy_i',\ldots,y_n\}$ to $\bar{c}_U(x_1,\ldots,x_ix_i'$, $\ldots,x_n)$. Since this map is additive and $\{y_1,\ldots,y_iy_i',\ldots,y_n\} = \{y_1,\ldots,y_i,\ldots,$ $y_n\} + \{y_1,\ldots,y_i',\ldots,y_n\}$ we get the result that $\bar{c}_U$ is also a multilinear function. ■

We now turn to perfect groups.

(7.7) LEMMA. *If G is a perfect group and $U \to G$ is an n-central extension then $\gamma_n(U)$ is also perfect.*

PROOF. First we show that $\gamma_{n+1}(U) = \gamma_n(U)$. From (7.6) we know that $\bar{c}_U(x_1,\ldots,x_n)$, as a function of x_n, say (i.e., fixing $x_1,\ldots,x_{n-1}$), is a homomorphism from G to the abelian group $\gamma_n(U)/\gamma_{n+1}(U)$. But G is perfect, so this map is trivial and so $\bar{c}_U(x_1,\ldots,x_n) = 1$ for all $x_1,\ldots,x_n$. As the elements $[\bar{c}_U(x_1,\ldots,x_n)]$ generate $\gamma_n(U)/\gamma_{n+1}(U)$, this group is trivial, i.e., $\gamma_n(U) = \gamma_{n+1}(U)$.

Thus, to show that $\gamma_n(U) = [\gamma_n(U),\gamma_n(U)]$ it is enough to prove that every $(n+1)$-commutator $[u_1,\ldots,u_{n+1}] \in \gamma_{n+1}(U)$ is in $[\gamma_n(U),\gamma_n(U)]$. From (2.4) we know that $[u_1,\ldots,u_{n+1}]$ only depends on the images of the u_i's in G. As

$\gamma_n(G) = G$ there exists $v \in \gamma_n(U)$ whose image in G is the same as the image of u_{n+1}. But this means that

$$[u_1,\ldots,u_{n+1}] = \big[[u_1,\ldots,u_n],v\big] \in \big[\gamma_n(U),\gamma_n(U)\big].$$

This completes the proof. ∎

(7.8) PROPOSITION. *Let G be a perfect group. Then* (i) *for every projective n-central extension $U \to G$, $n \geq 2$, the extension $\gamma_n(U) \to G$ is the universal central extension.* (ii) *For every $n \geq m \geq 2$, the natural morphism $\Phi^n_m : \Gamma_n(G) \to \Gamma_m(G)$ is an isomorphism. In particular,* $\mathrm{gr}_n \Gamma(G) = 0$ *for every $n \geq 1$.*

PROOF. It is shown in Milnor's book [9, p. 44] that a central extension $V \to G$ is universal iff it is projective and V is perfect. By (7.7), if $U \to G$ is n-central and projective then $\gamma_n(U)$ is perfect. We know that $\gamma_n(U) \to G$ is a central extension from (4.2). To show that $\gamma_n(U) \to G$ is projective let $W \to G$ be a central extension. Then it is also n-central ($n \geq 2$) and a lift $U \to W$ of id_G exists. Its restriction to $\gamma_n(U)$ is a lift of id_G from $\gamma_n(U)$ to W. This proves (i). The extensions $\Gamma_n(G) = \gamma_n(U) \to G$ and $\Gamma_m(G) \to G$ are both universal central extensions and it is easily checked that Φ^n_m is the unique isomorphism between them. This proves (ii). ∎

8. Lie structure

Associated to the lower central series filtration is the graded abelian group $\mathrm{gr}\,\gamma(G) = \bigoplus_{n=1}^{\infty} \mathrm{gr}_n\,\gamma(G)$ and this group has a natural Lie algebra structure (over $\mathbf{Z}$). As $\Gamma_n(G)$ is an extension of $\gamma_n(G)$, there is an associated surjection $\mathrm{gr}_n\,\Gamma(G) \to \mathrm{gr}_n\,\gamma(G)$ of abelian groups, so we would like $\mathrm{gr}\,\Gamma(G) = \bigoplus_{n=1}^{\infty} \mathrm{gr}_n\,\Gamma(G)$ to have a *natural* Lie structure too, which would render the map $\mathrm{gr}\,\Gamma(G) \to \mathrm{gr}\,\gamma(G)$ a morphism of Lie algebras, and so that everything will be functorial.

(8.1) THEOREM. (i) *For $n, m \geq 1$ there is a unique map $\{\ ,\ \} : gr_n\,\Gamma(G) \otimes_{\mathbf{Z}} \mathrm{gr}_m\,\Gamma(G) \to \mathrm{gr}_{n+m}\,\Gamma(G)$ which, if $m = 1$, sends $\{x_1,\ldots,x_n\} \otimes \{y\}$ to $\{x_1,\ldots,x_n,y\}$. This map satisfies the Jacobi identity $\big\{\{a,b\},c\big\} = \big\{\{a,c\},b\big\} + \big\{a,\{b,c\}\big\}$, so that* $\mathrm{gr}\,\Gamma$ *with $\{\ ,\ \}$ is a Lie algebra.*

(ii) *This Lie structure is natural, i.e. $\{\ ,\ \} : \mathrm{gr}\,\Gamma \otimes \mathrm{gr}\,\Gamma$ is a morphism of functors.*

(iii) *The map* $\mathrm{gr}\,\Gamma(G) \to \mathrm{gr}\,\gamma(G)$ *is a natural surjective Lie algebra morphism.*

PROOF. (i) The uniqueness of such a map is clear from the formula $\big\{\{x_1,\ldots,x_n\},\{y\}\big\} = \{x_1,\ldots,x_n,y\}$. To prove existence, let $a \in \mathrm{gr}_n\,\Gamma(G)$, $b \in \mathrm{gr}_m\,\Gamma(G)$, let $\alpha \in \Gamma_n(G)$ and $\beta \in \Gamma_m(G)$ represent a, b, respectively, and let $U \to G$, $V \to G$ be projective n- and m-central extensions, respectively, such that $\alpha \in \gamma_n(U)$, $\beta \in \gamma_m(V)$. We need to take the commutator of α and β. Let $W \to G$ be a projective $(n+m)$-central extension. As $U \to G$ is also an $(n+m)$-central extension, there is a lifting $f\colon W \to U$ of id_G and, similarly, a lifting $W \to V$. By (2.6), the induced map $\gamma_n(W) \to \gamma_n(U)$ is surjective. But, in fact, more is true. If N is the kernel of $W \to G$ then $f([N,(n-1)W]) = 1$ and if $\bar{f}$ is the induced map $W/[N,(n-1)W] \to U$ then, by (3.5) and the discussion in (4.1), it induces an isomorphism

$$\gamma_n(W/[N,(n-1)W]) = \gamma_n(W)/[N,(n-1)W] \overset{\approx}{\to} \gamma_n(U).$$

This means that not only is $\gamma_n(W) \to \gamma_n(U)$ surjective, but that the pre-image of an element of $\gamma_n(U)$ is unique modulo $[N,(n-1)W]$.

So let $\alpha_1 \in \gamma_n(W)$ represent α and $\beta_1 \in \gamma_m(W)$ represent β. By (2.1), $[\alpha_1, \beta_1]$ is in $\gamma_{n+m}(W)$. We define $\{a,b\}$ to be the class of $[\alpha_1,\beta_1]$ modulo $\gamma_{n+m+1}(W)$. The proof that this is independent of all the choices made is a repeated application of the identity

$$(*) \qquad [x,yz] = [x,y]\,[y,[x,z]]\,[x,z], \qquad [xy,z] = [y,z]\,[[z,y],x]\,[x,z],$$

which was already used before. For example, let us prove that the choice of β does not change the result. If $\beta' \in \gamma_m(V)$ is another representative for b then $\beta' = \tau\beta$, where $\tau \in \gamma_{m+1}(V)$. A pre-image β_1', for β', in $\gamma_m(W)$ must equal $\tau_1\beta_1$, where $\tau_1 \in [N,(m-1)W]\cdot\gamma_{m+1}(W)$. Indeed, by our considerations above modulo $[N,(m-1)W]$ there is a unique choice of pre-image for τ, and this choice is in $\gamma_{m+1}(W)$. Write $\tau_1 = \tau_1'\cdot\tau_1''$ with $\tau_1' \in [N,(n-1)W]$, $\tau_1'' \in \gamma_{m+1}(W)$. Then

$$[\alpha_1,\tau_1\beta_1] = [\alpha_1,\tau_1]\cdot[\tau_1,[\alpha_1,\beta_1]]\cdot[\alpha_1,\beta_1],$$

$$[\alpha_1,\tau_1] = [\alpha_1,\tau_1'\cdot\tau_1''] = [\alpha_1,\tau_1']\cdot[\tau_1',[\alpha_1,\tau_1'']]\cdot[\alpha_1,\tau_1''].$$

Clearly, $[\tau_1,[\alpha_1,\beta_1]] \in [N,\gamma_{n+m}(W)] = \{1\}$. From (2.1) and (1.2) it follows that $[\alpha_1,\tau_1'] = 1$. Similarly, $[\tau_1',[\alpha_1,\tau_1'']] = 1$ and $[\alpha_1,\tau_1''] \in \gamma_{n+m+1}(W)$ by (2.1). Summing up, we see that $[\alpha_1,\tau_1\beta_1] = [\alpha_1,\beta_1]$ modulo $\gamma_{n+m+1}(V)$. This proves that our definition of $\{a,b\}$ is independent of one choice made. The rest of the verifications are similar (and as tedious) and will be omitted.

It is easily verified that $\{\{x_1,\ldots,x_n\},\{y\}\} = \{x_1,\ldots,x_n,y\}$: if $z_1,\ldots,z_n,z$ represent $x_1,\ldots,x_n,y$, respectively, in W then, by our definition, $\{\{x_1,\ldots,x_n\},\{y\}\}$ is represented by $[[x_1,\ldots,x_n],y]$ whose class in $\mathrm{gr}_{n+1}\Gamma(G)$ is just $\{x_1,\ldots,x_n,y\}$.

If $\{\ ,\ \}$ is to be a homomorphism from $\mathrm{gr}_n\Gamma(G) \otimes \mathrm{gr}_m\Gamma(G)$ we must prove that it is bilinear. Let us show, for example, that $\{a + a', b\} = \{a,b\} + \{a',b\}$. With U, V, W as above let $\alpha, \alpha' \in \gamma_n(U)$, $\beta \in \gamma_m(V)$ be representatives of a, a', b, respectively, and $\alpha_1, \alpha_1' \in \gamma_n(W)$, $\beta_1 \in \gamma_m(W)$ representatives of α, α', β, respectively. We have to prove that

$$[\alpha_1 \cdot \alpha_1', \beta_1] = [\alpha_1, \beta_1] \cdot [\alpha_1', \beta_1] \bmod \gamma_{n+m+1}(W).$$

This follows, again, from (*) for

$$[\alpha_1 \cdot \alpha_1', \beta_1] = [\alpha_1', \beta_1]\,[[\beta_1, \alpha_1'], \alpha_1]\,[\alpha_1, \beta_1]$$

and $[[\beta_1, \alpha_1'], \alpha_1] \in \gamma_{n+m+1}(W)$ because $n, m \geq 1$, while $[\alpha_1', \beta_1]$ and $[\alpha_1, \beta_1]$ commute modulo $\gamma_{n+m+1}(W)$.

To prove the Jacobi identity for $\{\ ,\ \}$ we use the following identity of Hall: Given a group X and three elements $x \in \gamma_n(X)$, $y \in \gamma_m(X)$, $z \in \gamma_l(X)$, then

$$[[x,y],z] \cdot [[y,z],x] \cdot [[z,x],y] = 1 \bmod \gamma_{n+m+l+1}(X).$$

Let $U \to G$, $V \to G$, $W \to G$, $Y \to G$ be projective n-, m-, l-, $(n + m + 1)$-central extensions, respectively. To compute something like $\{\{a,b\},c\}$ with $a \in \mathrm{gr}_n \Gamma(G)$, $b \in \mathrm{gr}_m \Gamma(G)$, $c \in \mathrm{gr}_l \Gamma(G)$, we can proceed by taking representatives $x \in \gamma_n(U)$ for a, $y \in \gamma_m(V)$ for b and $z \in \gamma_l(W)$ for c and $x_1 \in \gamma_n(Y)$, $y_1 \in \gamma_m(Y)$, $z_1 \in \gamma_l(Y)$ that map is x, y, z, respectively, under the maps $Y \to U$, $Y \to V$, $Y \to W$ that exist by projectivity. Then it is easy to see that $\{\{a,b\},c\}$ is represented by $[[x_1,y_1],z_1]$. Similarly, $\{\{b,c\},a\}$ is represented by $[[y_1,z_1]x_1]$ and $\{\{c,a\},b\}$ by $[[z_1,x_1],y_1]$. Thus the Jacobi identity clearly follows from Hall's.

(ii) The naturality to be proved means that, for every homomorphism $f: G \to H$, the diagram

$$\begin{array}{ccc} \mathrm{gr}\,\Gamma(G) \otimes \mathrm{gr}\,\Gamma(G) & \xrightarrow{\{\ ,\ \}} & \mathrm{gr}\,\Gamma(G) \\ {\scriptstyle \mathrm{gr}\,\Gamma(f) \otimes \mathrm{gr}\,\Gamma(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}\,\Gamma(f)} \\ \mathrm{gr}\,\Gamma(H) \otimes \mathrm{gr}\,\Gamma(H) & \xrightarrow{\{\ ,\ \}} & \mathrm{gr}\,\Gamma(H) \end{array}$$

commutes, i.e., $\mathrm{gr}\,\Gamma(f)$ is a homomorphism of Lie algebras. This is proved conveniently by a judicious choice of representatives. If $a \in \mathrm{gr}_n \Gamma(G)$, $b \in \mathrm{gr}_m \Gamma(G)$ and $x \in \Gamma_n(G)$, $y \in \Gamma_m(G)$ represent a, b, respectively, it is clear that $\Gamma_n(f)(x)$ represents $\mathrm{gr}_n \Gamma(f)(a)$, etc., so the proof follows by a straightforward checking of definitions.

(iii) This is also a simple follow-up of definitions. ■

9. Computation of $\operatorname{gr}\Gamma(G)$

The explicit computation of $\operatorname{gr}\Gamma(G)$ is achieved through two results. The first is the computation when G is abelian, the second is that the abelianization map $G \to G_{ab}$ induces an isomorphism on $\operatorname{gr}\Gamma$.

It will be useful to introduce the concept of 'Lie algebra of a module'. If k is a commutative ring and M is a k-module then

(9.1) Definition. The Lie algebra of M (over k) is a morphism of k-modules $M \to \mathfrak{L}(M)$, where $\mathfrak{L}(M)$ is a k-Lie algebra, which is universal, i.e., if $M \to L$ is a k-module map of M into a k-Lie algebra L then there is a unique map $\mathfrak{L}(M) \to L$, of k-Lie algebras, such that the diagram

$$\begin{array}{ccc} & \nearrow & \mathfrak{L}(M) \\ M & & \downarrow \\ & \searrow & L \end{array}$$

is commutative.

The existence of this construction is shown as follows. First, if M is a free module we can take $\mathfrak{L}(M)$ to be the Lie-algebra generated by $M = T_1(M)$ in the tensor algebra $T(M) = \bigoplus_{n=0}^{\infty} T_n(M)$, and the "identity" map as the map from M to $\mathfrak{L}(M)$. If M is not free, there is still a Lie algebra generated by $M = T_1(M)$ in $T(M)$, but to avoid possible complications we can do the following. Let $0 \to K \to E \to M \to 0$ be a *free* presentation for M, i.e., the sequence is an exact sequence of k-modules and E is a free k-module. In $\mathfrak{L}(E)$ we identify E as $\mathfrak{L}_1(E)$, i.e. $\mathfrak{L}(E) = \sum_{n=1}^{\infty} \mathfrak{L}_n(E)$ where $\mathfrak{L}_n(E) = \mathfrak{L}(E) \cap T_n(E)$. Then let $I(K)$ be the Lie-ideal generated, in $\mathfrak{L}(E)$, by $K \subset E$. Clearly, $I(K)$ is a graded ideal and $I_1(K) = K$, so $(\mathfrak{L}(E)/I(K))_1 = M$. This gives a k-module map from M to $\mathfrak{L}(E)/I(K)$ and we claim that this arrow has the properties required of the arrow $M \to \mathfrak{L}(M)$. We leave the easy verification to the reader. Note that one can prove directly, using the 'Schanuel lemma', that the correspondence $M \to \mathfrak{L}(E)/I(K)$ is independent of the presentation and is actually functorial.

In the case of interest to us, $k = \mathbf{Z}$ and the module is a finitely generated abelian group G. If $G \approx \mathbf{Z}/m_1\mathbf{Z} \times \cdots \times \mathbf{Z}/m_r\mathbf{Z}$ then it is clear the $\mathfrak{L}(G)$ is the free Lie algebra on r-generators, say $b_1, \ldots, b_r$, subject only to the relations $m_1 b_1 = 0, \ldots, m_r b_r = 0$.

(9.2) Theorem. *If G is a finitely generated abelian group then* $\operatorname{gr}\Gamma(G) \approx \mathfrak{L}(G)$ *as graded Lie algebras.*

PROOF. As $\mathrm{gr}_1\Gamma(G)$ generates $\mathrm{gr}\,\Gamma(G)$ by (7.1), the 'identity' map $\mathfrak{L}_1(G) = G \to \mathrm{gr}_1\Gamma(G)$ extends to a Lie algebra surjection $\mathfrak{L}(G) \to \mathrm{gr}\,\Gamma(G)$, and this map is of course graded. We will show that it is an isomorphism by exhibiting, for each $n \geq 1$, a homomorphism $\mathrm{gr}_n\Gamma(G) \to \mathfrak{L}_n(G)$ which is left inverse to $\mathfrak{L}_n \to \mathrm{gr}_n\Gamma$. This would clearly suffice. To define it, we recall the definition of Γ_n and $\mathrm{gr}_n\Gamma$ in terms of a free presentation.

Let $1 \to R \to F \to G \to 1$ be such. Then $\mathrm{gr}_n\Gamma(G) = \gamma_n(F)/\gamma_{n+1}(F)\cdot[R,(n-1)F]$. We know that $\bigoplus_{n=1}^{\infty}\gamma_n(F)/\gamma_{n+1}(F) = \mathrm{gr}\,\gamma(F)$ is a free Lie algebra and is, in fact, $\mathfrak{L}(F_{ab})$; see Serre's book [15, p. 4.10]. Thus there is a surjection of graded Lie algebras $\mathrm{gr}\,\gamma(F) \to \mathfrak{L}(G)$. It will be convenient to use the presentation $F \to G$ where, if $G \approx \mathbf{Z}/m_1\mathbf{Z} \times \cdots \times \mathbf{Z}/m_r\mathbf{Z}$, then F is free on r generators $e_1,\ldots,e_r$ each mapping to a generator of the corresponding factor of G. Then R is generated by $[F,F] \cup \{e_1^{m_1},\ldots,e_r^{m_r}\}$. We will show now that the map $\mathrm{gr}_n\gamma(F) \to \mathfrak{L}_n(G)$ just constructed vanishes on $[R,(n-1)F]\cdot\gamma_{n+1}(F)/\gamma_{n+1}(F)$. Let us denote by $\{x_1,\ldots,x_n\}_F$ the class in $\mathrm{gr}_n\gamma(F)$ of the commutator $[x_1,\ldots,x_n]$. It is known (and discussed above, see §7 and §8) that $\{\ldots\}_F$ is multilinear and the value of $\{x_1,\ldots,x_n\}$ only depends on the classes of the x_i's in F_{ab}. Clearly, $[R,(n-1)F]\cdot\gamma_{n+1}(F)/\gamma_{n+1}(F)$ is generated by the elements $\{x_1,\ldots,x_n\}_F$ with $x_1 \in R$. If $x_1 \in [F,F]$ then $[x_1,\ldots,x_n] \in \gamma_{n+1}(F)$ and $\{x_1,\ldots,x_n\}_F = 0$. If $x_1 = e_j^{m_j}$ then $\{x_1,\ldots,x_n\}_F = m_j\cdot\{e_j,x_2,\ldots,x_n\}_F$ and we need to show that the map $\mathrm{gr}_n\gamma(F) \to \mathfrak{L}_n(G)$ vanishes on $m_j\{e_j,x_2,\ldots,x_n\}_F$. If $\bar{x} \in G$ denotes the image of $x \in F$ we need to show that the n-fold Lie product $[[\cdots[\bar{e}_j,\bar{x}_2,],\bar{x}_3]\cdots,\bar{x}_n]$ in $\mathfrak{L}_n(G)$ is annihilated by m_j. But this is true as $\bar{e}_j$ is a generator of the j-th factor, $\mathbf{Z}/m_j\mathbf{Z}$, of G, and thus of order m_j. This proves that there exists a homomorphism $\mathrm{gr}_n\Gamma(G) \to \mathfrak{L}_n(G)$ which is onto and which takes $\{y_1,\ldots,y_n\} \in \mathrm{gr}_n\Gamma(G)$ to the n-fold Lie product $[\cdots[y_1,y_2],\cdots,y_n]$ in $\mathfrak{L}_n(G)$. Since these n-fold Lie products generate $\mathfrak{L}_n(G)$ (as an abelian group) we see that the composition $\mathfrak{L}_n(G) \to \mathrm{gr}_n\Gamma(G) \to \mathfrak{L}_n(G)$ is the identity, because it is the identity on the set of these n-fold products.

This completes the proof. ∎

(9.3) THEOREM. *If G is a finitely generated group then the abelianization map $G \to G_{ab}$ induces an isomorphism* $\mathrm{gr}\,\Gamma(G) \to \mathrm{gr}\,\Gamma(G_{ab})$.

PROOF. As $G \to G_{ab}$ is onto and $\mathrm{gr}\,\Gamma(G_{ab})$ is generated by $\mathrm{gr}_1\Gamma(G_{ab})$, we see that $\mathrm{gr}\,\Gamma(G) \to \mathrm{gr}\,\Gamma(G_{ab})$ is onto. To construct a Lie algebra map in the opposite direction, note that $\mathrm{gr}_1\Gamma(G)$ is also G_{ab}, so the module map $\mathfrak{L}_1(G_{ab}) = G_{ab} \xrightarrow{\mathrm{id}} G_{ab} = \mathrm{gr}_1\Gamma(G) \subset \mathrm{gr}\,\Gamma(G)$ extends to a Lie algebra map $\mathfrak{L}(G_{ab}) \to \mathrm{gr}\,\Gamma(G)$. It is easy to see that the composition $\mathrm{gr}\,\Gamma(G) \to \mathrm{gr}\,\Gamma(G_{ab}) = \mathfrak{L}(G_{ab}) \to \mathrm{gr}\,\Gamma(G)$

takes $\{x_1, \ldots, x_n\} \in \mathrm{gr}_n \Gamma(G)$ to itself. Thus it is the identity, and the theorem is proved. ∎

10. Applications

We know, from (7.5), that if G is a nilpotent group of nilpotence c then for every $n \geq 1$ the canonical filtration of $\Gamma_n(G)$ ends in $\{1\}$, after at most c steps. Thus if G is finite we see that $|\Gamma_n(G)|$ divides the product $|\mathrm{gr}_n \Gamma(G)| \cdot |\mathrm{gr}_{n+1} \Gamma(G)| \cdots |\mathrm{gr}_{n+c-1} \Gamma(G)|$. Similarly, when G is not finite, if we want to estimate the rank of $\Gamma_n(G)$ (i.e., the number $\sum_{j=1}^{\infty} \dim_{\mathbf{Q}} \left(\mathbf{Q} \otimes \gamma_j(\Gamma_n(G))/\gamma_{j+1}(\Gamma_n(G))\right)$) then we can say that

$$\operatorname{rank} \Gamma_n(G) \leq \sum_{j=0}^{c-1} \operatorname{rank}\left(\mathrm{gr}_{n+j} \Gamma(G)\right).$$

The groups $\mathrm{gr}_m \Gamma(G)$ we 'know' quite well: $\mathrm{gr}_m \Gamma(G) \approx \mathcal{L}_m(G_{ab})$ by Theorem (9.2). Following the notation of [15], if L_X is the free Lie algebra (over $\mathbf{Z}$) on a set X of cardinality d then the group of elements of degree m in L_X is a free abelian group of rank $l_d(m)$ and there is a wonderful formula for $l_d(n)$:

$$l_d(n) = \frac{1}{n} \sum_{m \mid n} \mu(m) d^{n/m} \qquad (\mu(m) \text{ is the Möbius function}).$$

Let us start with the rank estimation.

(10.1) Proposition. *Let G be a nilpotent group of class c and let* $\operatorname{rank}(G_{ab}) = d$. *Then for every* $n \geq 1$, $\operatorname{rank}(\Gamma_n(G)) \leq l_d(n) + l_d(n+1) + \cdots + l_d(n+c-1)$.

Proof. The canonical filtration of $\Gamma_n(G)$ is

$$\Gamma_n(G) \supset \operatorname{im} \Gamma_{n+1}(G) \supset \cdots \supset \operatorname{im} \Gamma_{n+c}(G) = \{1\}.$$

Each quotient in this sequence is a homomorphic image of the corresponding $\mathrm{gr}_* \Gamma(G)$. So its rank is dominated by the rank of the corresponding $\mathrm{gr}_* \Gamma(G)$. The rank of $\mathrm{gr}_m \Gamma(G)$ is precisely $l_d(m)$, which proves the result. ∎

The case $n = 2$ is of interest. Then we have the exact sequence

$$0 \to H_2(G, \mathbf{Z}) \to \Gamma_2(G) \to \gamma_2(G) \to 1.$$

This gives the

(10.2) Corollary. *If G is nilpotent of class c and* $\operatorname{rank}(G_{ab}) = d$ *then* $\operatorname{rank}(H_2(G)) \leq l_d(2) + \cdots + l_d(c+1) - \operatorname{rank}(\gamma_2(G))$. ∎

This result is 'best possible' in the sense that it is an *equality* for free nilpotent groups, i.e., if $G = F/\gamma_{c+1}(F)$ with F free.

For example, if G is nilpotent of class 2, we get

$$\operatorname{rank}(H_2(G)) \le \frac{1}{2}(d^2 - d) + \frac{1}{3}(d^3 - d) - \operatorname{rank}(G').$$

This is a new kind of estimate and being 'best possible' should be useful.

We now turn to estimating the order of $\Gamma_n(G)$ when G is a finite nilpotent group. We denote by $\lambda_n(G)$ the order of $\mathfrak{L}_n(G_{ab})$. It is possible to write explicit formulas for $\lambda_n(G)$ in terms of the function $l_d(m)$, but these formulas are involved and we will settle for a numerical example below. As mentioned in the introduction, the variation of these functions (relative to G) may be of interest. The result is

(10.3) PROPOSITION. *If G is a finite nilpotent group of class c then, for every $n \ge 1$, $|\Gamma_n(G)|$ divides the product $\lambda_n(G)\cdot\lambda_{n+1}(G)\cdots\lambda_{n+c-1}(G)$.* ■

As before, the case $n = 2$ is of special interest. The result is that $|H_2(G,\mathbf{Z})|$ *divides* $\lambda_2(G)\cdots\lambda_{c+1}(G)/|G'|$.

Let us write out this result in the case $c = 2$. Say G is a p-group such that G_{ab} is of order p^a and G' is of order p^b. It can be proved (by a somewhat involved induction) that of all abelian groups H of order p^a, $|\mathfrak{L}_n(H)|$ is maximal when H is elementary abelian. Hence we can estimate $\lambda_n(G)$ by $p^{l_a(n)}$. Then our result is: $|H_2(G)|$ divides $p^{l_a(2)+l_a(3)-b}$. But $l_a(2) = \frac{1}{2}(a^2 - a)$, $l_a(3) = \frac{1}{3}(a^3 - a)$, so

$$|H_2(G)| \quad \text{divides} \quad p^{(a^2-a)/2+(a^3-a)/3-b}.$$

This can be compared to the estimate (assuming $G' = Z(G)$)

$$\begin{aligned} |H_2(G)| &\le |G'|^{a-1}\cdot|H_2(G_{ab})| \qquad \text{(see [5])} \\ &\le p^{b(a-1)+a(a-1)/2}. \end{aligned}$$

We see that the dependence on b, for example, is quite different in the two kinds of estimates.

POSTSCRIPT. After this paper was submitted for publication, I learnt of the interesting paper [4] by Blackburn and Evens, in which bounds on the number of generators of $H_2(G,\mathbf{Z})$ which are similar to those of (10.3) (in the sense that the 'Witt numbers' $l_a(n)$ occur in them) are obtained. They also show that their bounds are best possible. But it seems that their results do not cover ours and that the Lie algebra approach of this paper is quite different from theirs.

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